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**On the Properties of Multiple-Valued
Functions that are Symmetric in Both
Variable Values and Labels**

by

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December 1997

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Prepared for: Department of Electrical and Computer Engineering
Naval Postgraduate School
Monterey, California 93943

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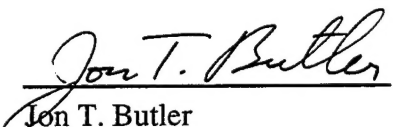
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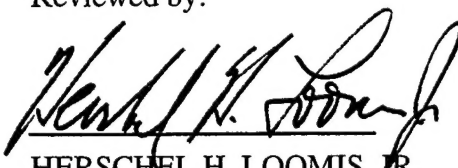
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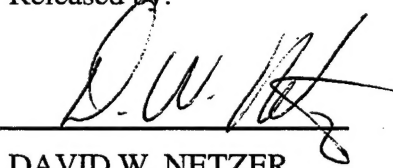
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REPORT DOCUMENTATION PAGEForm Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington DC 20503.

1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE December 2, 1997		3. REPORT TYPE AND DATES COVERED Technical Report	
4. TITLE AND SUBTITLE On the Properties of Multiple-Valued Functions that are Symmetric in Both Variable Values and Labels				5. FUNDING NUMBERS N/A	
6. AUTHOR(S) Jon T. Butler and Tsutomu Sasao					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Department of Electrical and Computer Engineering Naval Postgraduate School Monterey, CA 93943-5000				8. PERFORMING ORGANIZATION REPORT NUMBER NPS-EC-97-015	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Department of Electrical and Computer Engineering Naval Postgraduate School Monterey, CA 93943-5000				10. SPONSORING/MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES The views expressed in this report are those of the author and do not reflect the official policy or position of the Department of Defense or the United States Government.					
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.				12b. DISTRIBUTION CODE A	
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14. SUBJECT TERMS Multiple-valued logic, benchmark function, computer-aided design				15. NUMBER OF PAGES 21	
				16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT SAR		

On the properties of multiple-valued functions that are symmetric in both variable values and labels

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Abstract:

Functions that are symmetric in both variable labels and variable values are important for use as benchmarks. We present the properties of such functions, showing that they are isomorphic to partitions on n (the number of variables) with no part greater than r (the number of logic values). From this, we do an enumeration. Further, we derive lower bounds, upper bounds, and exact values for the number of prime implicants in the minimal sum-of-products expressions for certain subclasses of these functions.

1. Introduction

A variable/value symmetric function remains unchanged if

1. the variables are permuted and/or
2. the logic values of all variables are permuted.

In the binary case, for example, the Exclusive OR function, $f(x_1, x_2) = x_1 \oplus x_2$, is symmetric. Thus, if the variables are interchanged, the function remains unchanged. This can be expressed as $f(x_1, x_2) = x_2 \oplus x_1 = x_1 \oplus x_2$. If the logic values of all variables are interchanged, the function is also unchanged. This can be expressed as $f(x_1, x_2) = \bar{x}_1 \oplus \bar{x}_2 = x_1 \oplus x_2$. Thus, the Exclusive OR function is variable/value symmetric.

We are motivated to study *multiple-valued* variable/value symmetric functions because of the discovery [7] of a class of *two-valued* variable/value symmetric functions that defeat simplification algorithms which produce irredundant sum-of-products expressions. Indeed, the Minato-Morreale [3,4] algorithm does the worst that it can do on such functions. That is, it produces the largest number of prime implicants in irredundant sum-of-products expressions for specific variable/value symmetric functions. The problem has a significant practical implication. The ratio of the number of prime implicants in functions derived from variable/value symmetric functions for the worst case is an arbitrarily large constant times the number of prime implicants in the best case, as n , the number of variables increases without bound. Thus, the Minato-Morreale algorithm may not only do poorly; it can do *very* poorly. This suggests the importance of understanding

such functions, since most algorithms, like the Minato-Morreale algorithm, produce irredundant sum-of-products expressions.

Variable/value symmetric functions have another significant practical implication. They are easy to generate (by computer), and they are tractably analyzed. Yet, they are difficult for current algorithms to minimize. Thus, they are ideal benchmark functions [6].

2. Background

Definition 2.1: $f(x_1, x_2, \dots, x_n)$ is a *variable symmetric function* iff $f(x_1, x_2, \dots, x_n)$ is unchanged by a permutation of the variables.

Definition 2.2: $f(x_1, x_2, \dots, x_n)$ is a *value symmetric function* iff $f(x_1, x_2, \dots, x_n)$ is unchanged by a permutation of the logic values of *all* variables.

Definition 2.3: $f(x_1, x_2, \dots, x_n)$ is a *variable/value symmetric function* iff $f(x_1, x_2, \dots, x_n)$ is a variable symmetric function and a value symmetric function.

There is a long history of work on variable symmetric functions (also called *symmetric* functions or *totally symmetric* functions), including multiple-valued variable symmetric functions [2,5,8]. We know of no formal study of variable/value symmetric functions.

Example 2.1: The constant functions are trivial variable/value symmetric functions. The truth table below shows two nontrivial variable symmetric functions on two three-valued variables. Of these, f_2 is variable/value symmetric.

Table 2.1. Examples of 2-variable 3-valued variable/value symmetric functions.

x_1	x_2	f_1	f_2
0	0	1	2
0	1	2	1
0	2	1	1
1	0	2	1
1	1	0	2
1	2	1	1
2	0	1	1
2	1	1	1
2	2	2	2

Observation 2.1: A symmetric function is characterized by a set S of *alpha* vectors each of the form $[\alpha_0, \alpha_1, \dots, \alpha_{r-1}]$, where α_i is the number of variables that have logic

value i . Specifically, for *all* assignments of values to the variables that correspond to a single alpha vector in S , the function has one value, 0, 1, ..., or $r-1$.

Example 2.2: The alpha vectors of the functions in Table 2.1 are shown in Table 2.2 below.

Table 2.2. Alpha vectors of the functions shown in Table 2.1.

α_0	α_1	α_2	f_1	f_2
2	0	0	1	2
1	1	0	2	1
1	0	1	1	1
0	2	0	0	2
0	1	1	1	1
0	0	2	2	2

Observation 2.2: Since $\alpha_0 + \alpha_1 + \dots + \alpha_{r-1} = n$, each alpha vector corresponds to an *r-part composition* on integer n , where each part is non-negative. For each alpha vector, the function value can be chosen in r ways. Thus, the number of symmetric functions is $r^{\binom{n+r-1}{r-1}}$, where $\binom{n+r-1}{r-1}$ is the number of compositions on n into r parts, where each part is 0, 1, 2, Specifically, a composition can be chosen by selecting r objects from n with repetition. This can be done in $\binom{n+r-1}{r-1}$ ways.

Observation 2.3: Since a variable/value symmetric function is a symmetric function in which the variable logic values can be permuted without changing the function, a variable/value symmetric function corresponds to an *r-part partition* on integer n , where each part is 0, 1, 2, For each partition, the function value can be chosen in r ways. Thus, the number of variable/value symmetric functions is $r^{P(n,r)}$, where $P(n,r)$ is the number of partitions on n into r parts.

Observation 2.4: A variable/value symmetric function is characterized by a set S of *beta* vectors each of the form $[\beta_0, \beta_1, \dots, \beta_{r-1}]$, where β_i is the number of variables that have logic value i , and $\beta_i \geq \beta_{i-1} \geq 0$. Specifically, for all assignments of values to the variables that correspond to a single beta vector or any permutation of the vector, the function has one value, 0, 1, ..., or $r-1$. Note that the index, i , of β_i does not represent a specific logic value, as in the case of alpha vectors.

Example 2.3: The beta vectors of the function f_2 in Table 2.2 are shown in Table 2.3 below.

Table 2.3. Beta vectors of the function f_2 shown in Table 2.2.

β_0	β_1	β_2	f_2
2	0	0	2
1	1	0	1

In this example, the function is 2 if two of the variables are 0, two are 1, or two are 2. Further, the function is 1 if one variable is 0 and one is 1, one variable is 0 and one is 2, one variable is 1 and one is 2. We can represent the beta vectors of Table 2.3 by a Ferrar's Graph as shown in Fig. 2.1 below. Here, a part of size i in the beta vector is represented as a line of i (large) dots.



Figure 2.1. Ferrar's diagram of the beta vectors of the function f_2 shown in Table 2.1.

Fig. 2.1 shows that a beta vector is a partition of n into r or fewer parts. Note that a Ferrar's Graph rotated 90 degrees is also a Ferrar's Graph. This shows that a partition of n into r or fewer parts corresponds to a partition of n with no part greater than r . This allows us to make the following statement.

Lemma 2.1: The number of n -variable r -valued variable/value symmetric functions is $r^{g_{n,r}}$, where $g_{n,r}$ is the number of partitions on n with no part greater than r .

It should be noted that $r^{g_{n,r}}$ includes functions on fewer than n variables, since one choice among those enumerated is all logic values the same; i.e. a constant function. We can calculate $g_{n,r}$, as follows. Let $G_r(x)$ be the ordinary generating function for the number of partitions on n with no part greater than r . Specifically,

$$G_r(x) = g_{0,r} + g_{1,r}x + g_{2,r}x^2 + \dots + g_{n,r}x^n + \dots,$$

where $g_{n,r}$ is the number of partitions on n with no part greater than r . By Lemma 1, we can express $G_r(x)$ as

$$G_r(x) = (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots) \dots (1 + x^r + x^{2r} + \dots),$$

where each polynomial factor represents the number of ways the parts, 1, 2, ... and r , of various values can occur. From this, we can express $G_r(x)$ as

$$G_r(x) = \frac{1}{(1-x)(1-x^2)\dots(1-x^r)}.$$

Using a polynomial package (in our case, MACSYMA), we can derive $G_r(x)$ for various r . For example,

$$G_2(x) = 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + 5x^9 + 6x^{10} + 6x^{11} + 7x^{12} + \dots$$

$$G_3(x) = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 10x^8 + 12x^9 + 14x^{10} + 16x^{11} + 19x^{12} + \dots$$

$$G_4(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + 15x^8 + 18x^9 + 23x^{10} + 27x^{11} + 34x^{12} + \dots$$

$$G_5(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 10x^6 + 13x^7 + 18x^8 + 23x^9 + 30x^{10} + 37x^{11} + 47x^{12} + \dots$$

$$G_6(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 14x^7 + 20x^8 + 26x^9 + 35x^{10} + 44x^{11} + 58x^{12} + \dots$$

3. Properties of variable/value symmetric functions

Lemma 3.1: A value symmetric function on two r -valued variables is variable symmetric, for $r \geq 2$.

Proof: Since there are two variables, there are two possibilities; the variable values are the same or they are different. If they are the same, interchanging them leaves the function unchanged. If they are different, since it is value symmetric, any permutation of the logic values leaves the function unchanged, including permutations that interchange the two values. Since interchanging the two values leaves the function unchanged, it is variable symmetric.

Q.E.D.

Lemma 3.1 does not extend to value symmetric functions on three or more variables. For example, $f(x_1, x_2, x_3) = \bar{x}_1 \bar{x}_2 x_3 \vee x_1 x_2 \bar{x}_3$ is value symmetric but not variable symmetric. From the observation in the proof of Lemma 3.1, we can conclude the following.

Corollary 3.1: The number of two variable r -valued functions that are value symmetric (and variable symmetric) is r^2 for all $r \geq 2$.

Alternatively, one can show that the coefficient of x^2 in $G_r(x)$ is 2 for all $r \geq 2$.

We now consider a special type of variable/value symmetric function.

Definition 3.1: Let $ST_r(n,k)$ be an n -variable function that is 1 when there are at least k variables that are 0, k that are 1, ..., and k that are $r-1$. $ST_r(n,k) = 0$ otherwise. Specifically, it is symmetric because its value depends only on the number of variables that are 0 and 1. $ST_r(n,k)$ is variable/value symmetric because the function specification is the same for all logic values; i.e. there is symmetry among the logic values.

Example 3.1: When $r = 2$, $ST_r(n,k)$ is a two valued logic function that is 1 iff there are one or more 0's and one or more 1's among the variable values. Algebraically, this can be expressed as $ST_2(3,1) = (x_1^0 \vee x_2^0 \vee \dots \vee x_n^0)(x_1^1 \vee x_2^1 \vee \dots \vee x_n^1)$. In the case of $ST_3(3,1)$, where the variables are ternary, $ST_3(3,1)$ is the OR of six minterms, in which each minterm represents a way to permute 0, 1, and 2.

Example 3.2: The beta vectors for $ST_2(6,1)$, $ST_2(6,2)$, and $ST_2(6,3)$ are shown in Table 3.4 below. The beta vectors for $ST_3(6,1)$ and $ST_3(6,2)$ are shown in Table 3.5 below.

Table 3.4. Beta vectors for $ST_2(6,1)$, $ST_2(6,2)$, and $ST_2(6,3)$.

β_0	β_1	$ST_2(6,1)$ logic value	$ST_2(6,2)$ logic value	$ST_2(6,3)$ logic value	$ST_2(6,1)$ # minterms	$ST_2(6,2)$ # minterms	$ST_2(6,3)$ # minterms	$ST_2(6,1)$ #minterms covered/#PIs covering	$ST_2(6,2)$ #minterms covered/#PIs covering	$ST_2(6,3)$ #minterms covered/#PIs covering
6	0	0	0	0	0	0	0	-	-	-
5	1	1	0	0	12	0	0	2/5	-	-
4	2	1	1	0	30	30	0	8/8	2/6	-
3	3	1	1	1	20	20	20	6/9	2/9	1/1
Total					62	50	20	16/	4/	1/

Table 3.5. Beta vectors for $ST_3(6,1)$ and $ST_3(6,2)$.

β_0	β_1	β_2	$ST_3(6,1)$ logic value	$ST_3(6,2)$ logic value	$ST_3(6,1)$ # minterms	$ST_3(6,2)$ # minterms	$ST_3(6,1)$ #minterms covered/#PIs covering	$ST_3(6,2)$ #minterms covered/#PIs covering
6	0	0	0	0	0	0	-	-
5	1	0	0	0	0	0	-	-
4	2	0	0	0	0	0	-	-
4	1	1	1	0	90	0	3/4	-
3	3	0	0	0	0	0	-	-
3	2	1	1	0	360	0	18/6	-
2	2	2	1	1	90	90	6/8	1/1
Total					540	90	27/	1/

Tables 3.4 and 3.5 also show the number of minterms associated with each beta vector. The totals of the number of minterms is shown, as well. Thus, for $ST_2(6,1)$, $ST_2(6,2)$, and $ST_2(6,3)$, the total number of minterms covered is 62, 50, and 20, respectively (out of 64). For $ST_3(6,1)$, and $ST_3(6,2)$, the total number of minterms covered is 540 and 90, respectively (out of 729).

Also shown in Tables 3.4 and 3.5 are the number of minterms each prime implicant covers of the type of minterm shown and the number of prime implicants that cover a specific single minterm of the type shown. So, for example, the minterms of $ST_2(6,1)$, associated with beta vector [5,1] have the property that each prime implicant covers 2 minterms with that beta vector, and, conversely, each minterm with that beta vector is covered by 5 prime implicants. Thus, the corresponding entry is 2/5. We examine this in depth now.

Definition 3.2: Literal $x^S = 1$ if $x \in S$; otherwise $x^S = 0$.

Example 3.3: If x is a 3-valued variable, then $x^{\{0,2\}} = 1$ if x is 0 or 2 and $x^{\{0,2\}} = 0$ if x is 1.

For convenience, if S consists of exactly one element, then we omit $\{\}$. Thus, $x^{\{i\}}$ is written as x^i . Note that $x^{\{0,1,\dots,r-1\}}$ is a constant 1.

Definition 3.3: Let $f(x_1, x_2, \dots, x_n)$ be a two-valued function with r -valued variables. $P = x_1^{S_1} x_2^{S_2} \dots x_n^{S_n}$ is an *implicant* of f , if f is 1 whenever $x_1 \in S_1$, $x_2 \in S_2$, and $x_n \in S_n$. P is a *prime implicant* (PI) of f if P' is not an implicant of f , where P' is the same as P except some S_i is replaced by S_i' , where $S_i' \supset S_i$.

P is a prime implicant if all literal sets S_i are as large as they can be.

Example 3.4: $x_1^0 x_2^1 x_3^1 = \bar{x}_1 x_2 x_3$ is an implicant of $ST_2(3,1)$ because, when $x_1^0 x_2^1 x_3^1$ is 1 (when $x_1 = 0$, $x_2 = 1$, and $x_3 = 1$), $ST_2(3,1)$ is also 1. However, it is not a prime implicant, because $x_1^0 x_2^{\{0,1\}} x_3^1 = x_1^0 x_3^1$ is an implicant of $ST_2(3,1)$. However, $x_1^0 x_3^1$ is a prime implicant because enlarging any of its literal sets causes it not to be an implicant.

Observation 3.5: $x_{i_1}^0 x_{i_2}^1$ is a prime implicant of $ST_2(n,1)$, where $i_1, i_2 \in \{1, 2, \dots, n\}$ and $i_1 \neq i_2$. Also, $x_{i_1}^0 x_{i_2}^1 \dots x_{i_r}^{r-1}$ is a prime implicant of $ST_r(n,1)$, such that $i_1, i_2, \dots, i_r \in \{1, 2, \dots, n\}$ and all i_j are distinct. Further, $x_{i_1}^0 x_{i_2}^0 \dots x_{i_k}^0 x_{i_{k+1}}^1 x_{i_{k+2}}^1 \dots x_{i_{2k}}^1 \dots x_{i_{(r-1)k+1}}^{r-1} x_{i_{(r-1)k+2}}^{r-1} \dots x_{i_{rk}}^{r-1}$ is a prime implicant of $ST_r(n,k)$, such that $i_1, i_2, \dots, i_{rk} \in \{1, 2, \dots, n\}$ and all i_j are distinct.

Theorem 3.1: $ST_r(n, k)$ has

$$\frac{n!}{(k!)^r (n - kr)!}$$

prime implicants.

Proof: The prime implicants of $ST_r(n, k)$ have the form

$$x_{i_1}^0 x_{i_2}^0 \dots x_{i_k}^0 x_{i_{k+1}}^1 x_{i_{k+2}}^1 \dots x_{i_{2k}}^1 \dots x_{i_{(r-1)k+1}}^{r-1} x_{i_{(r-1)k+2}}^{r-1} \dots x_{i_{rk}}^{r-1}.$$

Specifically, this expression is 1 when all k variables in a specific set $(\{x_{i_1} x_{i_2} \dots x_{i_k}\})$ are 0, when all k variables in another specific set $(\{x_{i_{k+1}} x_{i_{k+2}} \dots x_{i_{2k}}\})$ are 1, ... and when all k variables in another specific set $(\{x_{i_{(r-1)k+1}} x_{i_{(r-1)k+2}} \dots x_{i_{rk}}\})$ are $r-1$. On the contrary, if some variable has fewer than k representatives, then the corresponding product term is 1 when fewer than k of those variables are 1, and thus the function has this property, contradicting the fact that the function is $ST_r(n, k)$. If some variable has more than k representatives, then deleting literals so there are exactly k representatives creates an implicant of the function that is implied by the original implicant; thus, the original implicant was not prime.

There are $n!/\{(k!)^r (n - kr)!\}$ implicants of this form. Specifically, each implicant corresponds to a permutation of $r+1$ objects, where r of the objects (the logic values) each occur as k identical objects and one object (variables not included) occur as $n - kr$ identical objects.

Q.E.D.

Example 3.5: Consider the functions in Tables 3.4 and 3.5. From Theorem 3.1, $ST_2(6, 1)$, $ST_2(6, 2)$, and $ST_2(6, 3)$ have a total of 30, 90, and 20 prime implicants, respectively, while $ST_3(6, 1)$, and $ST_3(6, 2)$ have a total of 120 and 90 prime implicants, respectively.

Lemma 3.2: Each prime implicant in $ST_r(n, k)$ covers r^{n-kr} minterms.

Proof: The prime implicants of $ST_r(n, k)$ have the form

$$x_{i_1}^0 x_{i_2}^0 \dots x_{i_k}^0 x_{i_{k+1}}^1 x_{i_{k+2}}^1 \dots x_{i_{2k}}^1 \dots x_{i_{(r-1)k+1}}^{r-1} x_{i_{(r-1)k+2}}^{r-1} \dots x_{i_{rk}}^{r-1}.$$

There are $n - rk$ variables not included, whose values can each be chosen in r ways. Each different specification represents a minterm covered by the prime implicant.

Q.E.D.

Example 3.6: Consider the functions in Tables 3.4 and 3.5. From Lemma 3.2, the prime implicants of $ST_2(6,1)$, $ST_2(6,2)$, $ST_2(6,3)$, $ST_3(6,1)$, and $ST_3(6,2)$ cover, $2^2=4$, $2^0=1$, $3^3=27$, and $3^0=1$ minterms, respectively. This is shown in Tables 3.4 and 3.5 in the last three column of the last row. So, for example, the $2^4 = 16$ minterms covered by prime implicants of $ST_2(6,1)$ are distributed to minterms associated with beta vectors $[5,1]$, $[4,2]$, and $[3,3]$ as 2, 8, and 6 respectively.

The following lemma represents an observations about sum-of-products expressions for $ST_r(n,k)$ functions that will be useful later when one already has one sum-of-products expression.

Lemma 3.3: Let F be a sum-of-products expression for $ST_r(n,k)$. Let F' be a sum-of-products expression derived from F by permuting all variable labels j in x_j^i according to permutation $\pi_{\text{variable}}: \{1,2,\dots,n\} \rightarrow \{1,2,\dots,n\}$ and by permuting all logic values i of x_j^i according to permutation $\pi_{\text{value}}: \{0,1,\dots,r-1\} \rightarrow \{0,1,\dots,r-1\}$. Then, F' is a sum-of-products expression for $ST_r(n,k)$.

Proof: The statement follows from the fact that $ST_r(n,k)$ is unchanged by a permutation of variable labels and logic values.

Q.E.D.

Lemma 3.3 simply states that, although $ST_r(n,k)$ is unchanged by a permutation of logic values and variable labels, a sum-of-products expression for $ST_r(n,k)$ may be changed.

Lemma 3.4: Every prime implicant of $ST_r(n,k)$ occurs in an MSOP of $ST_r(n,k)$.

Proof: Consider any prime implicant P of $ST_r(n,k)$. Consider a prime implicant P' in an MSOP of $ST_r(n,k)$. By a permutation of variable labels and logic values, we can convert P' into P . Applying this permutation to all prime implicants in the MSOP yields an MSOP that contains P .

Q.E.D.

We are interested in the number of prime implicants in the minimal sum-of-products expression (MSOP) expression of $ST_r(n,k)$. Tables 3.4 and 3.5 provide an insight on this. For example, in Table 3.4, we see that for $ST_2(6,1)$, there are 12 minterms associated with beta vector $[5,1]$ and (from the third column from the left) each prime implicant covers 2 minterms in this group. It follows that at least $\lceil 12/2 \rceil = 6$ prime implicants are needed in the MSOP of $ST_2(6,1)$. From beta vector $[4,2]$, at least $\lceil 30/8 \rceil = 4$ prime implicants are needed, and from beta vector $[3,3]$, at least $\lceil 20/6 \rceil = 4$ prime implicants are needed. Beta vector $[5,1]$ provides the greatest restriction, and we have a lower bound on the number of prime implicants for $ST_2(6,1)$ of 6 prime implicants. In a similar manner, we can calculate a lower bound on the number of prime implicants of

$ST_2(6,2)$, $ST_2(6,3)$, $ST_3(6,1)$, and $ST_3(6,2)$ as 15, 20, 30, and 90, respectively. In general, we can state

Theorem 3.2: A lower bound on the number of prime implicants in the MSOP of $ST_r(n,1)$ is

$$\frac{n!}{(n-r+1)!}$$

Proof: Consider a set S of minterms corresponding to beta vector $[n-r+1, 1, 1, \dots, 1]$. The number of minterms in S is $rn!/(n-r+1)!$, when $n > r$ and $n!$ when $n = r$. If $n > r$, each prime implicant, $x_i^0 x_{i_2}^1 \dots x_{i_r}^{r-1}$, covers r minterms in S , since there are r ways to choose the variable values not in $x_i^0 x_{i_2}^1 \dots x_{i_r}^{r-1}$. If $n = r$, there are no other values to specify. Thus, for both $n > r$ and $n = r$, at least $n!/(n-r+1)!$ prime implicants are needed.

Q.E.D.

Consider now an upper bound on the number of prime implicants in an MSOP for $ST_r(n,1)$. Any sum-of-products expression for $ST_r(n,1)$ has the form

$$x_n^0 F_0 \vee x_n^1 F_1 \vee \dots \vee x_n^{r-1} F_{r-1} \vee F, \quad (1)$$

where F_i and F are sum-of-products expressions on x_1, x_2, \dots, x_{n-1} . Each contains literals of the form x_j^k , where $0 \leq k \leq r-1$. In the case of F_i , $k \neq i$, while F is unrestricted. That is, both contain all variables except x_n . F contains literals of the form x_j^k , for all $i \in \{0, 1, \dots, r-1\}$, while F_i does not contain the literal x_j^i . For example, an MSOP for $ST_3(4,1)$ can be written as

$$ST_2(3,1) = x_1^0 x_3^1 \vee x_3^0 x_2^1 \vee x_2^0 x_1^1,$$

which can be verified by a Karnaugh Map. From Lemma 3.3, it follows that by permuting the logic values (in this case, complementation) or by permuting the variables, we can obtain another MSOP.

$$ST_2(3,1) = x_3^0 x_1^1 \vee x_2^0 x_3^1 \vee x_1^0 x_2^1.$$

As another example, an MSOP for $ST_3(4,1)$ can be written as

$$\begin{aligned} ST_3(4,1) = & x_4^0 (x_1^1 x_3^2 \vee x_3^1 x_2^2 \vee x_2^1 x_1^2) \vee x_4^1 (x_3^0 x_1^2 \vee x_2^0 x_3^2 \vee x_1^0 x_2^2) \\ & \vee x_4^2 (x_1^0 x_3^1 \vee x_3^0 x_2^1 \vee x_2^0 x_1^1) \vee (x_1^0 x_2^1 x_3^2 \vee x_2^0 x_3^1 x_1^2 \vee x_3^0 x_1^1 x_2^2) \end{aligned} \quad (2)$$

It is straightforward (but tedious) to verify that this is indeed is a sum-of-products expression for $ST_3(4,1)$. To verify that it is minimal requires some insight.

Lemma 3.5 If $x_n^0 F_0 \vee x_n^1 F_1 \vee \dots \vee x_n^{r-1} F_{r-1} \vee F$ is an MSOP of $ST_{r-1}(n-1,1)$, then

1. F_i is an MSOP of $ST_{r-1}(n-1,1)$ on logic values $\{0, 1, \dots, r-1\} - \{i\}$, and

2. F is an MSOP of $ST_r(n-1,1)$ with (consensus) terms removed that are also covered in $x_n^0 F_0 \vee x_n^1 F_1 \vee \dots \vee x_n^{r-1} F_{r-1}$.

Proof: When $x_n=i$, $ST_r(n,1) = F_i$, and F_i is $ST_{r-1}(n-1,1)$ on logic values $\{0,1,\dots,r-1\}-\{i\}$. The expression for F_i must be an MSOP of $ST_{r-1}(n-1,1)$; otherwise, it can be replaced by an MSOP, thus reducing the products in the expression for $ST_r(n,1)$, contradicting the statement that $ST_{r-1}(n-1,1)$ is an MSOP. Consider a minterm m covered by $ST_{r-1}(n-1,1)$ which has at least one 0, one 1, ..., and one $r-1$ among x_1, x_2, \dots , and x_{n-1} . Then, m is covered by a prime implicant that is in F or is a consensus term among $x_n^0 F_0 \vee x_n^1 F_1 \vee \dots \vee x_n^{r-1} F_{r-1}$ or both. Among all terms must be an MSOP for $ST_r(n-1,1)$; otherwise $x_n^0 F_0 \vee x_n^1 F_1 \vee \dots \vee x_n^{r-1} F_{r-1} \vee F$ is not an irredundant SOP for $ST_r(n,1)$. Property 2 follows from this observation.

Q.E.D.

If we add the consensus terms to F , forming F' , we obtain the following expression.

$$ST_3(4,1) = x_4^0(x_1^1 x_3^2 \vee x_3^1 x_2^2 \vee x_2^1 x_1^2) \vee x_4^1(x_3^0 x_1^2 \vee x_2^0 x_3^2 \vee x_1^0 x_2^2) \vee x_4^2(x_1^0 x_3^1 \vee x_3^0 x_2^1 \vee x_2^0 x_1^1) \vee (x_1^0 x_2^1 x_3^2 \vee x_2^0 x_3^1 x_1^2 \vee x_3^0 x_1^2 x_2^1 \vee x_2^1 x_3^2 x_1^0 \vee x_1^2 x_3^1 x_2^0 \vee x_3^2 x_2^1 x_1^0) \quad (3)$$

The last three terms are each redundant. F' is an MSOP for $ST_3(3,1)$; it is the OR of **minterms** representing all 6 ways to permute three logic values among three variables. In this expression, the F_i terms are minimal, but F' is not. Therefore, this expression would be an MSOP if F' has as few terms as possible, which means that the largest number of terms of F' should be covered by the F_i terms. However, no more than three such terms can be covered collectively by the F_i terms. It follows that the original expression (with the last three terms removed) is an MSOP.

The two conditions of Lemma 3.6 are necessary as the proof shows. They are not sufficient. That is, the irredundant sum-of-products expression for $ST_3(4,1)$ satisfies

$$ST_3(4,1) = x_4^0(x_1^1 x_3^2 \vee x_3^1 x_2^2 \vee x_2^1 x_1^2) \vee x_4^1(x_3^0 x_1^2 \vee x_2^0 x_3^2 \vee x_1^0 x_2^2) \vee x_4^2(x_1^0 x_3^1 \vee x_3^0 x_2^1 \vee x_2^0 x_1^1) \vee (x_1^0 x_2^1 x_3^2 \vee x_2^0 x_3^1 x_1^2 \vee x_3^0 x_1^2 x_2^1 \vee x_2^1 x_3^2 x_1^0 \vee x_1^2 x_3^1 x_2^0 \vee x_3^2 x_2^1 x_1^0) \quad (4)$$

the two conditions, but it is not minimal. This is the above equation with the F_1 term modified. Specifically, too few (none) product terms in F appear as consensus terms covered by the F_i terms. Fig. 3.1 below shows a graph representation of the F_i terms in both expressions. In this figure, an arc represents a prime implicant in an F_i term.

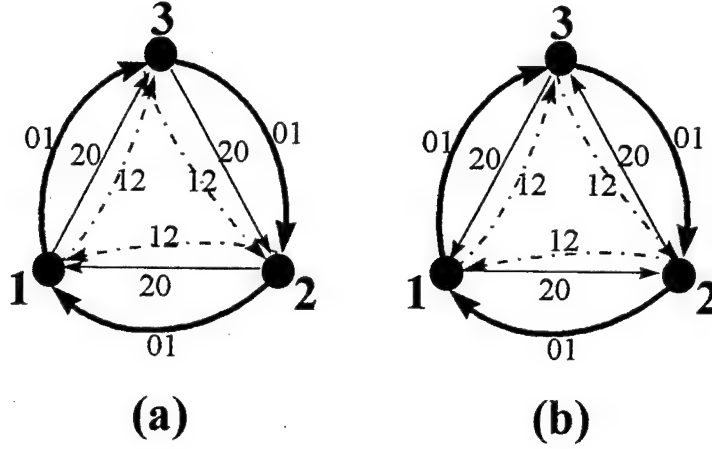


Figure 3.1. Graph representation of the F_i terms in (3) and ((4)).

For example, $x_3^0 x_2^1$ is represented as an arc labeled by the logic values 01 from node 3 representing x_3 to node 2, representing x_2 . The fact that $x_3^0 x_2^1 \vee x_2^0 x_1^1 \vee x_1^0 x_3^1$ is an MSOP of $ST_3(2,1)$ is seen in the graph by the cycle of arcs labeled 01 from x_3 to x_2 , x_2 to x_1 , and x_1 to x_3 . Similarly, the arcs labeled 12 form a cycle, as do the arcs labeled 20. A consensus term occurs also as a cycle. For example in Fig. 3.1a, there is a consensus term $x_1^0 x_3^1 x_2^2$ associated with the 01 arc from 1 to 3, the 12 arc from 3 to 2, and a 20 arc from 2 to 1. Notice in Fig. 3.1b, there are triples of arcs of this type and thus no consensus terms.

How can we produce an MSOPs for $ST_r(n,1)$?

Algorithm: Produce an MSOP for $ST_r(n,1)$ given an MSOP for $ST_r(n-1,1)$.

1. Set F of $ST_r(n,1)$ equal to the MSOP of $ST_r(n-1,1)$.
2. Form the largest set of prime implicants in F as consensus terms and form F_0, F_1, \dots, F_{r-1} . Remove these terms from F .

We illustrate by an example. Form an MSOP of $ST_3(5,1)$ from an MSOP of $ST_3(4,1)$ as follows. Specifically, use the MSOP shown in (2). Consider the consensus terms to remove. F_i will be an MSOP on 4 variables and 2 logic values. Fig. 3.2 below shows that we can create at most two consensus terms, $x_1^0 x_2^1 x_3^2 \vee x_2^0 x_3^1 x_1^2$. Indeed, this figure was generated by first drawing the largest number of consensus terms and then completing it so that the F_i terms formed MSOPs. The SOP so formed is

$$\begin{aligned}
 ST_3(4,1) = & x_5^0 (x_1^1 x_4^2 \vee x_4^1 x_3^2 \vee x_3^1 x_2^2 \vee x_2^1 x_1^2) \vee x_5^1 (x_1^0 x_2^2 \vee x_4^0 x_3^2 \vee x_2^0 x_4^2 \vee x_3^0 x_1^2) \\
 & \vee x_5^2 (x_1^0 x_4^1 \vee x_4^0 x_3^1 \vee x_3^0 x_2^1 \vee x_2^0 x_1^1) \\
 & \vee x_4^0 x_1^1 x_3^2 \vee x_4^0 x_3^1 x_2^2 \vee x_4^0 x_2^1 x_1^2 \vee x_4^1 x_3^0 x_1^2 \vee x_4^1 x_2^0 x_3^2 \vee x_4^1 x_1^0 x_2^2 \\
 & \vee x_4^2 x_1^0 x_3^1 \vee x_4^2 x_3^0 x_2^1 \vee x_4^2 x_2^0 x_1^1 \quad \vee \quad x_3^0 x_1^1 x_2^2
 \end{aligned}$$

This proves that the MSOP for $ST_3(5,1)$ has $3 \times 4 + 12 - 2 = 22$ prime implicants, which is two more than the lower bound given in Theorem 3.2.

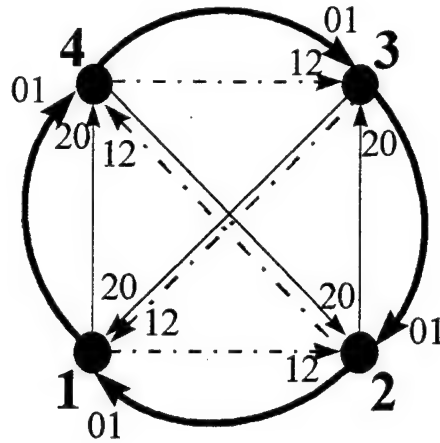


Figure 3.2. Graph representation of the F_i in the formation of an MSOP of $ST_3(5,1)$

For specific values of n and r ($= n-1$), we can determine the exact number of prime implicants in the MSOP. Figure 3.3 below shows a graph representation of the sum-of-products expression of $ST_2(3,1)$. Here, nodes represent prime implicants of $ST_2(3,1)$ and edges represent minterms covered by prime implicants. Specifically, each minterm is covered by two prime implicants (e.g. minterm 011 is covered by $0*1$ and $01*$), which correspond to the two nodes ($0*1$ and $01*$) incident to the edge (011). The problem of finding an MSOP is identical to the problem of finding a set S with the fewest nodes such that each edge is incident to at least one node in S . From Figure 3.3, it can be seen that there are two sets each with nodes that satisfy this criteria. These represent two MSOPs.

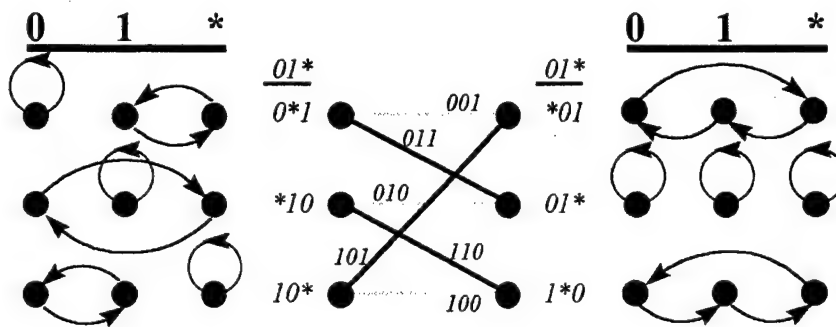


Figure 3.3. Graph representation of prime implicants and minterms for $ST_2(3,1)$.

Theorem 3.3: The number of prime implicants in an MSOP of $ST_{n-1}(n,1)$ is

$$\frac{n!}{2}.$$

Proof: Consider a graph $G(V,E)$, where V is the set of prime implicants of $ST_{n-1}(n,1)$ and E is the set of minterms of $ST_{n-1}(n,1)$. Each minterm m has the form $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$, where each logic value $\{0,1, \dots, n-2\}$ occurs as one i_j , except one (repeated) logic value that occurs twice. Let the repeated logic value be i_s and i_t . The two prime implicants that cover m have the form 1) $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$, where $x_s^{i_s}$ is omitted (i.e. replaced by $x_s^{\{0,1,\dots,n-2\}} = 1$) and 2) $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$, where $x_t^{i_t}$ is omitted (i.e. replaced by $x_t^{\{0,1,\dots,n-2\}} = 1$). Note that there are $n!$ prime implicants; each corresponds to a permutation on the elements $\{0,1, \dots, n-2, *\}$, where $0, 1, \dots, n-2$ are the $n-1$ logic values whose position determines the literal with that logic value and $*$ represents the missing variable.

We now show that G is bipartite. From this, it follows that the nodes in the part with the fewest nodes represents an MSOP of $ST_{n-1}(n,1)$. However, from Theorem 3.2, a lower bound on this number is $n!/2$. Since the total number of nodes is $n!$, and both parts have no less than $n!/2$ prime implicants, it follows that both have exactly this number. Therefore, there are two MSOPs of $ST_{n-1}(n,1)$ each with $n!/2$ prime implicants.

Consider a minterm m and the two prime implicants, P_1 and P_2 , that cover m . Let P_1 and P_2 have the representation shown below in Fig. 3.4. Specifically, P_1 is represented as $i_0 i_1 \dots i_{s-1} * i_{s+1} \dots i_{n-1}$, a compact representation of $x_1^{i_0} x_2^{i_1} \dots x_{s-1}^{i_{s-1}} * x_{s+1}^{i_{s+1}} \dots x_{n-1}^{i_{n-1}}$. Since $i_0 i_1 \dots i_{s-1} * i_{s+1} \dots i_{n-1}$ is a permutation of the elements $\{0,1, \dots, n-2, *\}$, it can be represented as a directed graph (the cycle

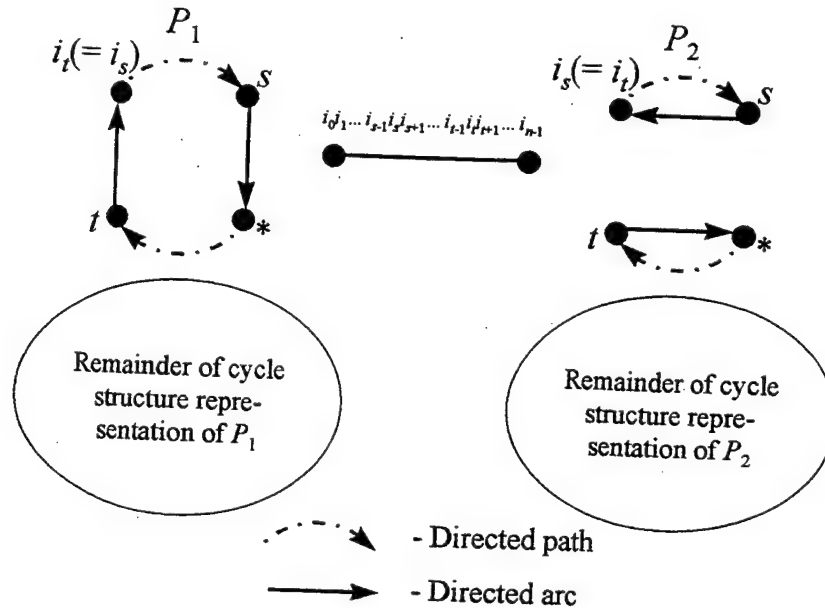


Figure 3.4. Cycle structure of prime implicants, P_1 and P_2 , which both cover the same minterm m .

structure representation) where the nodes correspond to elements of $\{0, 1, \dots, n-2, *\}$, and the arcs are $0 \rightarrow i_0, 1 \rightarrow i_1, \dots, * \rightarrow i_s, \dots$, and $n-1 \rightarrow i_{n-1}$ for $i_j \in \{0, 1, \dots, n-2, *\}$. Similarly, let P_1 be represented as $i_0 i_1 \dots i_{t-1} * i_{t+1} \dots i_{n-1}$. The minterm covered by P_1 and P_2 is therefore $i_0 i_1 \dots i_{s-1} i_s i_{s+1} \dots i_{t-1} i_t i_{t+1} \dots i_{n-1}$, where $i_s = i_t$. Consider the cycle structure representation of P_1 . There are two possibilities 1) $*$ and $i_s = i_t$ are in the same cycle or 2) $*$ and $i_s = i_t$ are in different cycles. In Fig. 3.4, the case where $*$ and $i_s = i_t$ are in the same cycle is shown. Note that the cycle structure of P_2 is the same as that of P_1 except that the arc from s to $*$ is replaced by an arc from s to $i_s = i_t$, while the arc from t to $i_s = i_t$ is replaced by an arc from t to $*$. As a result, the single cycle of P_1 becomes two cycles in P_2 . This is shown in Fig. 3.4. Conversely, if $*$ and $i_s = i_t$ are in different cycles in P_1 , it can be seen that these two cycles become one in P_2 . From this, it follows that G is bipartite, where one part corresponds to all permutations with an even number of cycles and the other part corresponds to all permutations with an odd number of cycles.

Q.E.D.

From the above discussion, we can state

Corollary 3.2: $ST_{n-1}(n, 1)$ has two MSOPs.

We can now prove the following

Lemma 3.7: An upper bound on the number of product terms in MSOP for $ST_r(n, 1)$ for $2 \leq r \leq 6$ is

Table 3.6. Upper bounds on the number of prime implicants in $ST_r(n, 1)$.

r	Upper Bound
2	n
3	$\frac{3}{2}n^2 - \frac{3}{2}n - 3$
4	$2n^3 - 6n^2 - 8n + 24$
5	$\frac{5}{2}n^4 - 15n^3 + \frac{5}{2}n^2 - 32n - 330$

Proof: Let $U(n, r)$ be an upper bound on the number of prime implicants in the MSOP of $ST_r(n, 1)$. From our previous observation on the decomposition $x_n^0 F_0 \vee x_n^1 F_1 \vee \dots \vee x_n^{r-1} F_{r-1} \vee F$ of $ST_r(n, 1)$, F_i must be an MSOP of $ST_{r-1}(n-1, 1)$, while F is an MSOP on $ST_r(n-1, 1)$ less consensus terms. Thus, a recursion relation for $U(n, r)$ is

$$U(n, r) = r U(n-1, r-1) + U(n-1, r).$$

Using $U(n,2) = n$ and $U(n,n)=n!$, one can recursively solve for the closed form upper bounds shown in Table 3.6.

Q.E.D.

From this result, we can state

Lemma 3.8: An upper bound on the number of product terms in an MSOP of $ST_r(n,1)$ is $O(\frac{r}{2}n^{r-1})$.

Proof: From Lemma 3.7, one can recursively solve for the coefficient of the largest term as $O(\frac{r}{2}n^{r-1})$.

Q.E.D.

Lemma 3.9: For fixed r , the number of product terms in an MSOP of $ST_r(n,1)$ is $O(n^{r-1})$.

Proof: Compare the upper and lower bounds for the number of product terms in an MSOP of $ST_r(n,1)$.

Q.E.D.

4. Experimental Results

Table 4.1 below shows the results of experiments on various classes of $ST_r(n,1)$ functions. Here, the number of prime implicants, as derived in Theorem 3.1 are shown in the second column labeled **No. of PIs**, the number of prime implicants calculated by a Quine-McCluskey type logic minimizer is shown in the third column labeled **QM**, and the lower bound derived in Theorem 3.2 in the fourth column labeled **LB**. The fifth column, labeled **UB**, shows an upper bound on the number of prime implicants as derived in Lemma 3.9. The last column, labeled **NWS**, shows the number of prime implicants obtained by a heuristic program designed to find an irredundant sum-of-products expression with many prime implicants.

Table 4.1 Number of prime implicants in total, as derived by Quine-McCluskey, lower and upper bounds on prime implicants in an MSOP, and in a large sum-of products.

Function	No. of PIs	QM	LB	UB	NSW
$ST_3(4,1)$	24	** 12	* 12	24	16
$ST_3(5,1)$	60	** 22	20	37.5	32
$ST_3(6,1)$	120	33	30	54	50
$ST_3(7,1)$	210	49	42	73.5	75
$ST_3(8,1)$	336	70	56	96	98
$ST_3(9,1)$	504	89	72	121.5	
$ST_4(5,1)$	120	** 60	* 60	250	96
$ST_4(6,1)$	360	159	120	432	210
$ST_4(7,1)$	840	282	210	686	
$ST_4(8,1)$	1,680	490	336	1024	
$ST_5(6,1)$	720	<i>a</i> 455	* 360	3240	600
$ST_5(7,1)$	2,520	<i>b</i> 1,231	832	6002.5	
$ST_6(7,1)$	5,040	3,365	* 2,520	50,421	4,320
$ST_7(8,1)$	40,320	32,760	*20,160	917,504	

* Known to be the exact number of prime implicants in an MSOP of $ST_r(n,1)$ by Theorem 3.3. ** Quine-McCluskey obtained a solution known to be an MSOP. *a* MINI2 obtained a 455 prime implicant solution. *b* MINI2 obtained a 1,206 prime implicant solution.

4. Concluding Remarks

Functions that are symmetric in both variable labels and variable values are important because they provide tractably analyzed functions that are difficult to minimize. Thus, they are useful as benchmark functions. Yet, as far as we know, there has been no formal study of such functions. In this paper, we have shown that there is a one to one correspondence between such functions and partitions on n with no part greater than r . We have shown how many prime implicants there are in each function, and, we have shown upper and lower bounds on the number of prime implicants in minimal sum-of-products expressions of a specific class of variable/value symmetric functions. For another class of variable/value symmetric functions, we have shown the exact number of prime implicants in minimal sum-of-products expressions.

5. Acknowledgments

This work was supported in part by a Grant in Aid for Scientific Research of the Ministry of Education, Science, Culture and Sports of Japan.

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